The University Of Sheffield.

Topological Recursion & Enumerative Geometry

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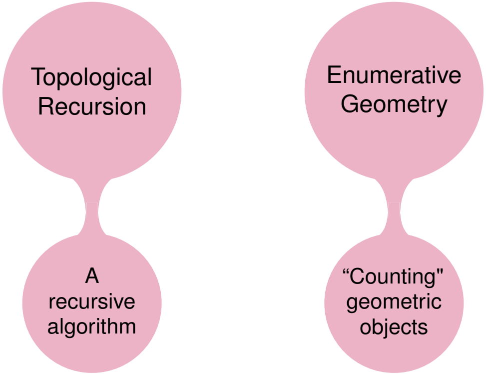
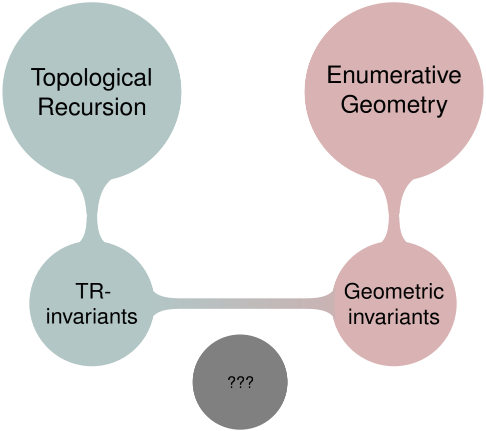
A tale of two concepts




Topological Recursion
Enumerative Geometry
TR-invariants
Geometric invariants
???

Topological Recursion (TR)

- First appeared in Random Matrix theory;
- Generates the asymptotic expansion of correlation functions of the model;
- Promoted to a universal mathematical theory in 2007 by Eynard and Orantin;
- Input: a spectral curve (initial conditions);
- Output: a sequence of multi-differentials $\{\omega_{g,n}\}$, where $g \geq 0$, and $n \geq 0$ with recursion on $2g - 2 + n$.

Input: a spectral curve

Definition

A *spectral curve*, $\mathcal{S}$, is a tuple (C, x, y, B), where

- C is a Riemann surface;
- x and y are meromorphic functions on C;
- $x : C \to \mathbb{P}^1$ is a ramified covering;
- B is a meromorphic differential on $C \times C$ with a double pole on the diagonal with bi-residue one.

Initial data: $\omega_{0,1}(z) = y(z)dx(z),$

$$\omega_{0,2}(z_1, z_2) = B(z_1, z_2).$$

Output

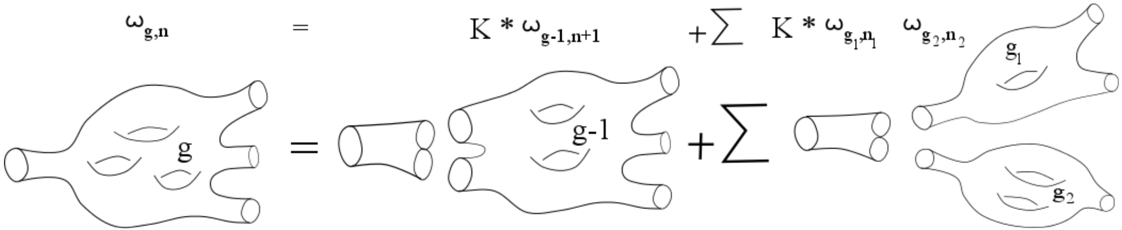
For $n \geq 0$, and $2g - 2 + n > 0$:

$$\omega_{g,n}(z_1, \cdots, z_n) := \sum_{i=1}^{r} \operatorname*{Res}_{z=a_i} \mathcal{K}(z_1, z)\Big(\omega_{g-1,n+2}(z, \sigma(z), z_2, \cdots, z_n) + \sum'_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\cdots,n\}}} \omega_{g_1,|I_1|+1}(z, z_{I_1})\omega_{g_2,|I_2|+1}(\sigma(z), z_{I_2})\Big),$$

where $\mathcal{K}$ is called the *recursion kernel*:

$$\mathcal{K}(z_1, z) := \frac{1}{2}\frac{\int_{w=\sigma(z)}^{z} \omega_{0,2}(z_1, w)}{(y(z) - y(\sigma(z))\mathrm{d}x(z)}.$$

Diagrammatic illustration of TR



Some examples

Step 1: 2g-2+n=1

- $\omega_{1,1}$
- $\omega_{0,3}$

$$\omega_{0,3}(z_1, z_2, z_3) = \sum_{i=1}^{r} \operatorname*{Res}_{z=a_i} \mathcal{K}(z_1, z)\left(\omega_{0,2}(z, z_2)\,\omega_{0,2}(\sigma(z), z_3) + \omega_{0,2}(z, z_3)\,\omega_{0,2}(\sigma(z), z_2)\right),$$

$$\omega_{1,1}(z_1) = \sum_{i=1}^{r} \operatorname*{Res}_{z=a_i} \mathcal{K}(z_1, z)\,\omega_{0,2}(z, \sigma(z)).$$

Enumerative geometry

Ancient questions, recent answers.

For example:

- Number of lines passing through 2 points in the plane
- Number of rational curves of degree d passing through $3d - 1$ points in the plane

General strategy:

1. Define and understand a moduli space of "all" geometric objects you're interested in;
2. Phrase geometric conditions in terms of subvarieties of the moduli space;
3. Invariants as intersection of subvarieties;

$\Longrightarrow$ counting things $\rightsquigarrow$ integration over moduli spaces

Relating the two: an example

Let

$$\mathcal{S} \equiv (C,\, x(z),\, y(z), \omega_{0,2}(z_1, z_2)) \;=\; \left(\mathbb{P}^1,\, \frac{z^2}{2},\, \frac{\sin(2\pi z)}{2\pi},\, \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\right),$$

with $\sigma(z) = -z$ and

$$\mathcal{K}(z_1, z) = \frac{1}{(z_1^2 - z^2)} \frac{\pi}{\sin(2\pi z)} \frac{\mathrm{d}z_1}{\mathrm{d}z},$$

Then,

$$\omega_{1,1}(z_1) = \operatorname*{Res}_{z=0} \frac{1}{(z^2 - z_1^2)} \frac{\pi}{\sin(2\pi z)} \frac{\mathrm{d}z_1}{\mathrm{d}z} \frac{\mathrm{d}z\,\mathrm{d}z}{(2z)^2} = \frac{1}{8z_1^2}\left(\frac{1}{z_1^2} + \frac{4\pi^2}{6}\right)\mathrm{d}z_1.$$

Mirzakhani recursion

The hyperbolic volume of the moduli space of genus 1 bordered Riemann surface with one geodesic of length L, is given by

$$\mathcal{V}_{1,1} = \frac{L^2 + 4\pi^2}{48}.$$

Taking the Laplace transform

$$\int_0^\infty \mathcal{V}_{1,1}(L) e^{-z_1 L} L \mathrm{d}L = \frac{1}{8z_1^4} + \frac{\pi^2}{12 z_1^2}.$$

$\implies$ equivalence under Laplace transform in general?

Example 2: Gromov-Witten theory of $\mathbb{P}^1$

Let

$$\mathcal{S} \equiv (C,\, x(z),\, y(z), \omega_{0,2}(z_1, z_2)) \;=\; \left(\mathbb{P}^1,\, z + \frac{1}{z},\, \log(z),\, \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right).$$

Then TR recovers the higher genus Gromov-Witten theory of $\mathbb{P}^1$.

Proved by P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz (2014) using Givental theory and Frobenius manifolds.

Relating the two continued

In some sense, expect the Laplace transform to play the role of mirror symmetry!

Some questions:

1. Given an enumerative problem, find spectral curve recovering it's invariants?
2. Given a spectral curve, does it exist a geometric interpretation?
3. Given an enumerative problem or TR spectral curve, does it exist an associated random matrix model?

Some possible strategies towards solutions:

- Frobenius manifolds/Givental theory;
- Integrable systems/Loop equations/Virasoro constraints;
- Brute force comparison;
- Physics.

Examples of recent and current research

- **BKMP conjecture** (now theorem): Open GW-theory of toric CY 3-folds (B. Eynard, N. Orantin, B. Fang, C.-C. M. Liu, Z. Zong);
- **Masur-Veech volumes** in the context of quadratic differentials (D. Chen, M. Möller, A. Souvaget, J.E. Andersen, G. Borot, V. Delacroix, S. Charbonnier, A. Giacchetto, D. Lewanski, C. Wheeler);
- "Normal", **double**, and **monotone Hurwitz numbers** (B. Eynard, M. Mulase, B. Safnuk, G. Borot, N. Do, M. Karev, R. Kramer, A. Popolitov, S. Shadrin);
- Enumeration of **hypermaps** recovered by TR (G. Carlet, J. van de Leur, H. Posthuma, S. Shadrin).

TR is also intimately connected to:
Cohomological (and topological) field theories, knot theory, quantum curves, the WKB approximation and BPS states.

My current work

Conjecture

Let

$$\mathcal{S} = \begin{cases} C = \mathbb{P}^1, \\ x = \dfrac{(-1)^k(1+(-1)^{n+1}z^{n+1})}{z^k}, \quad n \geq 1,\ 1 \geq k \geq n \\ y = log(z), \\ B(z_1, z_2) = \displaystyle\sum_{k \in \mathbb{Z}_n} \frac{(-1)^{\frac{2k}{n}} dz_1\ dz_2}{(z_1 - (-1)^{\frac{2k}{n}} z_2)^2}. \end{cases}$$

Then, TR recovers the higher genus Gromov-Witten theory of the orbifold $\mathbb{P}^1_{k,n+1-k}$.